A Review of Week1 Stuff: **Lesson 2**

The Method of Separation of Variables:

The posted lessons are part of the Differential Equations course that I taught at Montgomery College in Germantown Maryland.

The lessons are written according to *Differential Equations* , Third Edition, by Blanchard, Devaney, and Hall , Brooks/Cole as the text book adopted for the class.

For any questions, comments or objections

You may write to me at

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Page 34

24.

$\frac{dy}{dt} = \frac{t}{y-t^2y}$, $y(0) = 4$

$\frac{dy}{dt}=\frac{t}{y-t^2y}$

$\frac{dy}{dt}=\frac{t}{y(1-t^2)}$

$y\frac{dy}{dt}=\frac{t}{1-t^2}$

$ydy =\frac{t}{1-t^2}dt$

$\int ydy =\int \frac{t}{1-t^2}dt$

$1 - t^2= u \rightarrow -2tdt = du \rightarrow tdt = -\frac{1}{2}du$

$\frac{y^2}{2}= -\frac{1}{2} \int \frac{du}{u}$

$\frac{y^2}{2}= -\frac{1}{2} \ln|u|+C$

$\frac{y^2}{2}= -\frac{1}{2} \ln|1 - t^2|+C$

when $t = 0, y = 4$

$\frac{4^2}{2}= -\frac{1}{2} \ln|1 - 0|+C$

$8 = C$

$\frac{y^2}{2} = -\frac{1}{2}\ln|1 - t^2| + 8$

#38 on the page 35.

amount of hot sauce in the pot of chilli

Let y be the amount (in spoons) if hot sauce in the chille at any time t (in minutes)

$\frac{dy}{dt}$ = **loss of hot sauce in each cup of chilli**

$\frac{dy}{dt} = -\frac{y}{32}$

$\frac{dy}{y} = -\frac{1}{32}dt$

$\int \frac{dy}{y} = \int -\frac{1}{32}dt$

$\ln y = -(t/32) + C$

when $t = 0,\ y = 12$ **spoons**

$\ln 12 = 0 + C$

$C = \ln 12$

$\ln \mathbf{y} = -(t/32) + \ln \mathbf{12}$

$\ln \mathbf{y} - \ln \mathbf{12} = -(t/32)$

$\ln\left(\frac{y}{12}\right) = -(t/32)$

$\frac{y}{12} = \mathbf{e}^{-(t/32)}$

$\mathbf{y} = \mathbf{12e}^{-(t/32)}$

wanted only 4 teaspoons

we need the value of t **such that** $y = 4$

$\ln\left(\frac{4}{12}\right) = -(t/32)$

$-\frac{t}{32} = \ln\left(\frac{1}{3}\right)$

$t = -32\ln\left(\frac{1}{3}\right) = 35.15559325$ **minutes**

35 Cups

Section 1.3:

Examples of the Slope Fields

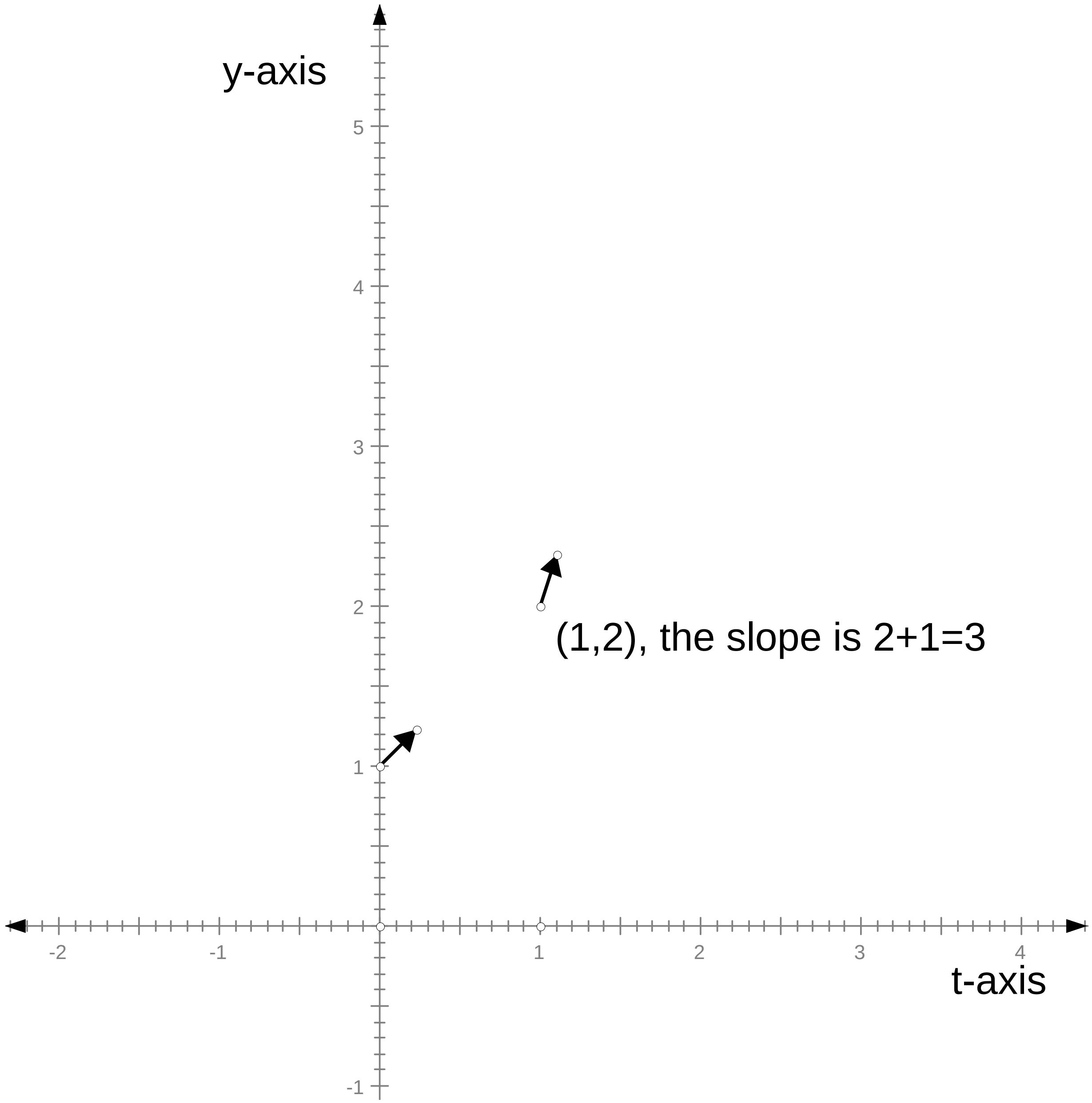
Consider the differential Equation

$\frac{dy}{dt} = \mathbf{y} + \mathbf{t}$

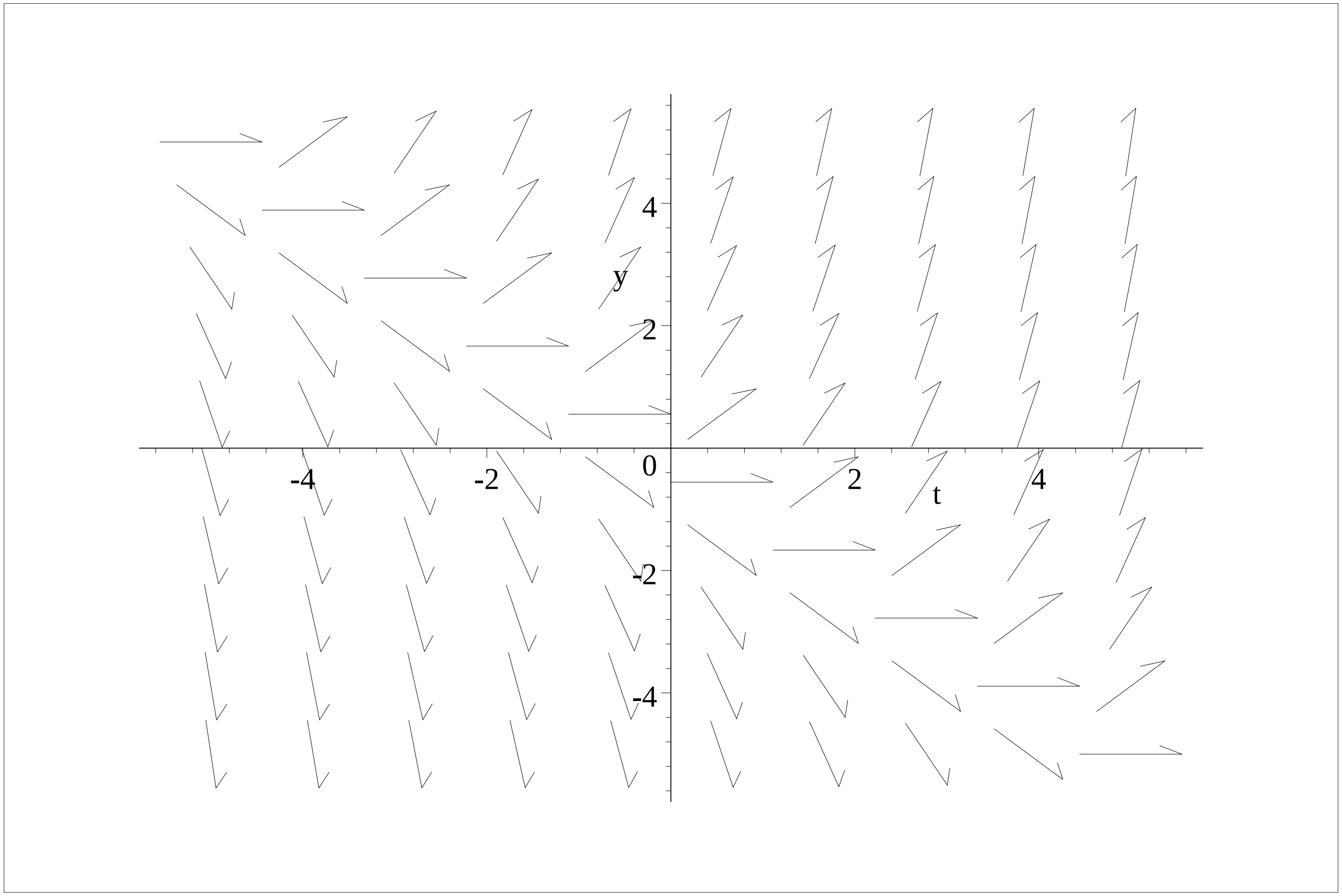
the slope of the tangent line to the solution curve at each point (t, y) **equals** $y + t$

A slope field for a differential equation can be obtained by drawing minilines that are tangent to the solution curve.

Drawing such a field only by hand


y-axis
5
4
3
2
1
-1
-2
-1
1
2
3
4
t-axis
(1,2), the slope is 2+1=3

Drawing such a field only by hand takes too much time, we can use computer programs to draw such slope fields



HPGSolver in the text is a good tool to draw such a slope field

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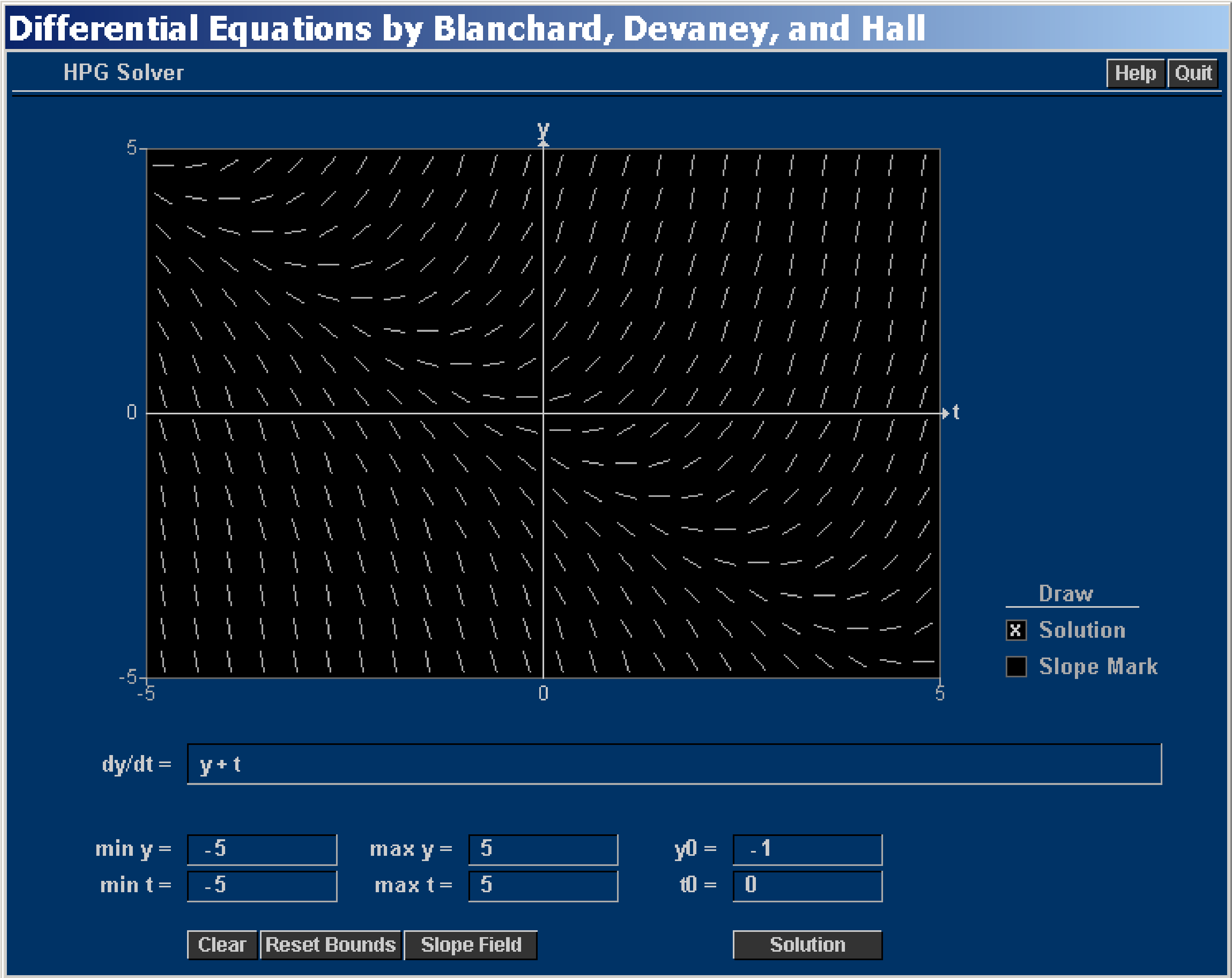
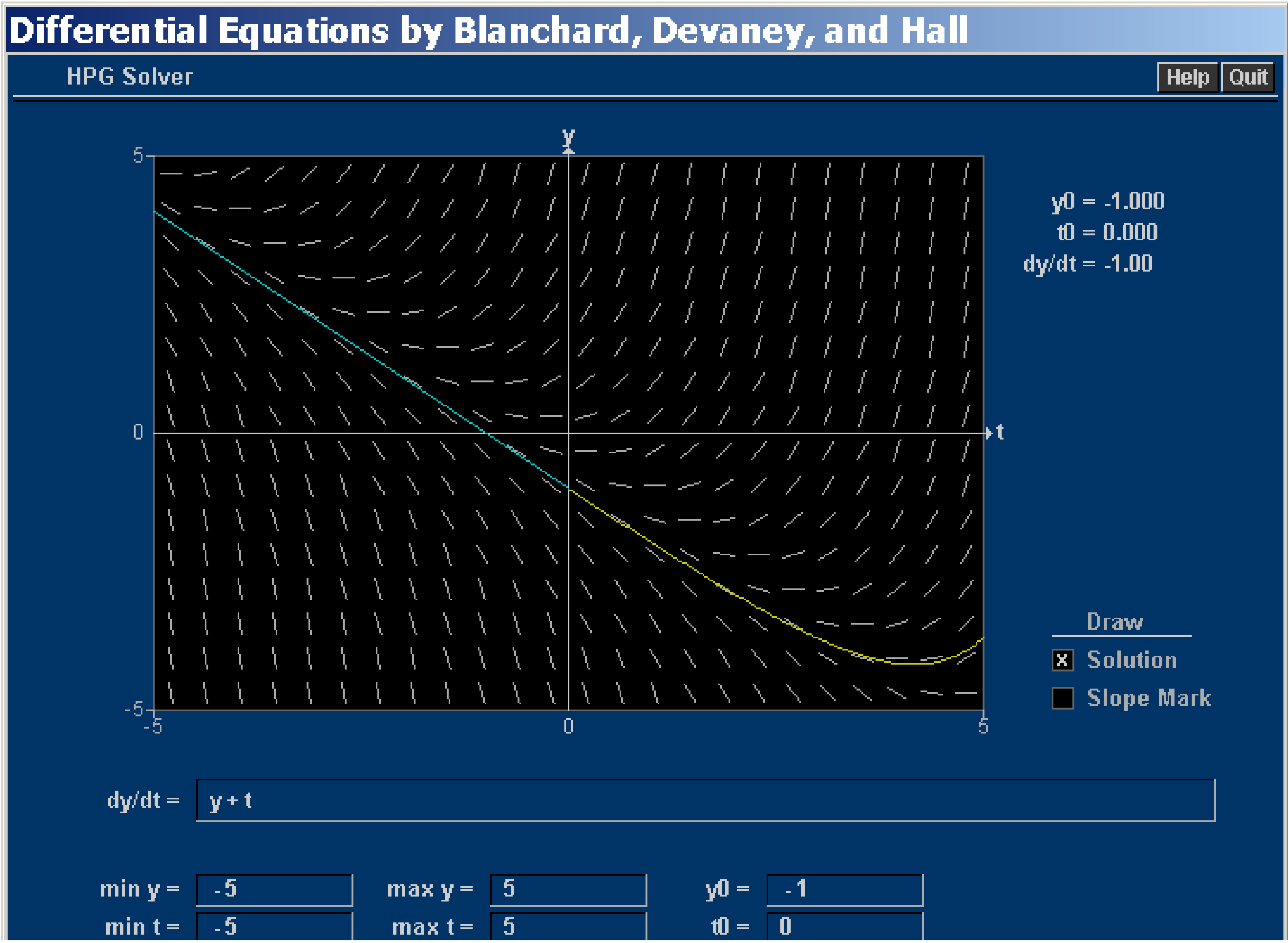
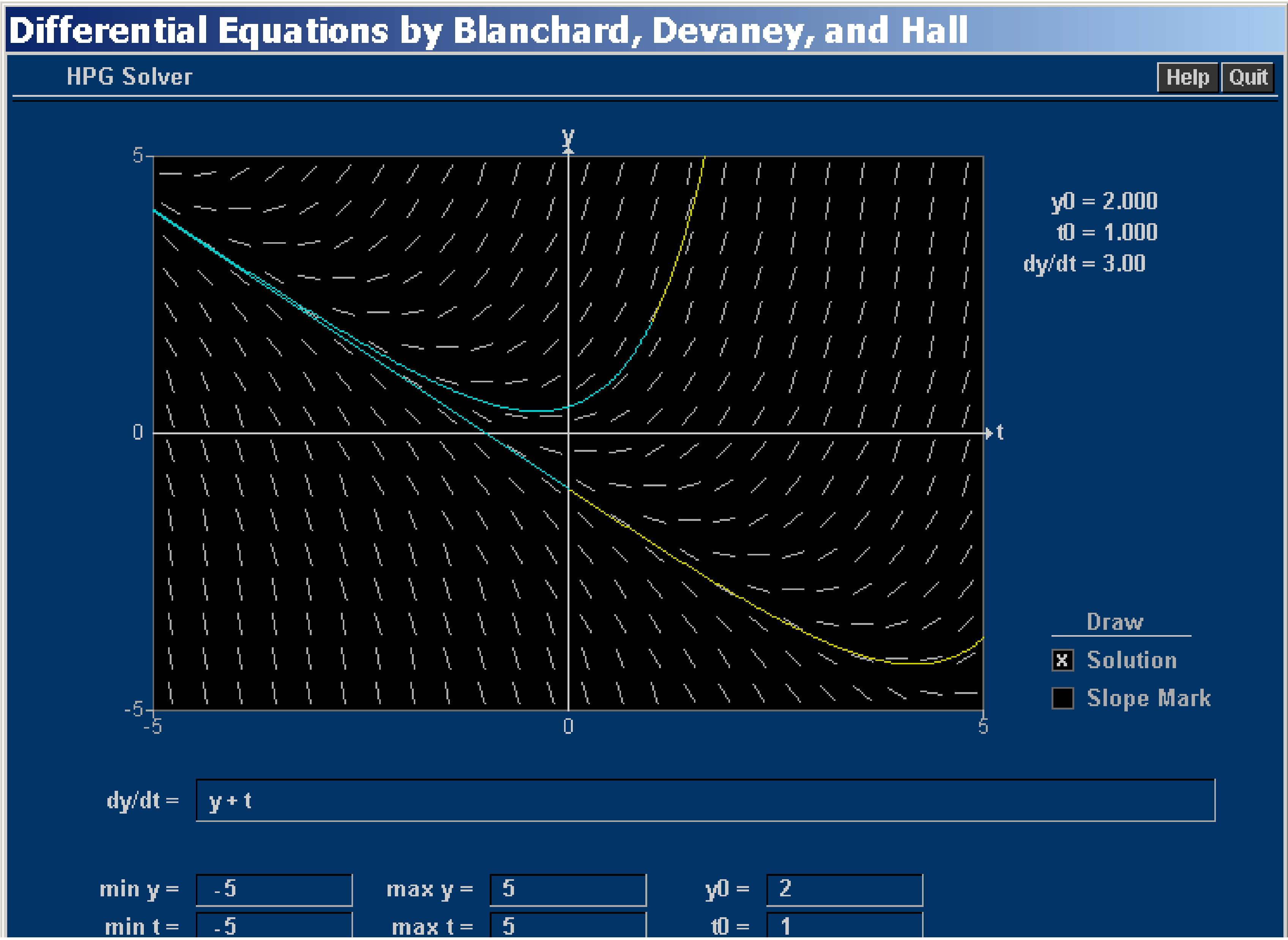

Differential Equations by Blanchard, Devaney, and Hall
HPG Solver
Help
Quit
y
t
5
0
-5
-5
0
5
Draw
Solution
Slope Mark
dy/dt =
y + t
min y =
-5
max y =
5
y0 =
-1
min t =
-5
max t =
5
t0 =
0
Clear
Reset Bounds
Slope Field
Solution

To look at the solution of the initial value problem $y(0)=-1$, we can use the HPG Solver in the way shown below



Differential Equations by Blanchard, Devaney, and Hall
HPG Solver
Help
Quit
y
5
0
-5
-5
0
5
t
y0 = -1.000
t0 = 0.000
dy/dt = -1.00
Draw
Solution
Slope Mark
dy/dt = y + t
min y = -5
max y = 5
y0 = -1
min t = -5
max t = 5
t0 = 0

Another solution curve may be obtained as shown below


Differential Equations by Blanchard, Devaney, and Hall
HPG Solver
Help
Quit
y
t
5
0
-5
-5
0
5
y0 = 2.000
t0 = 1.000
dy/dt = 3.00
Draw
Solution
Slope Mark
dy/dt =
y + t
min y =
-5
max y =
5
y0 =
2
min t =
-5
max t =
5
t0 =
1

For the differential Equation

$\frac{dy}{dt} = y + t$**,**

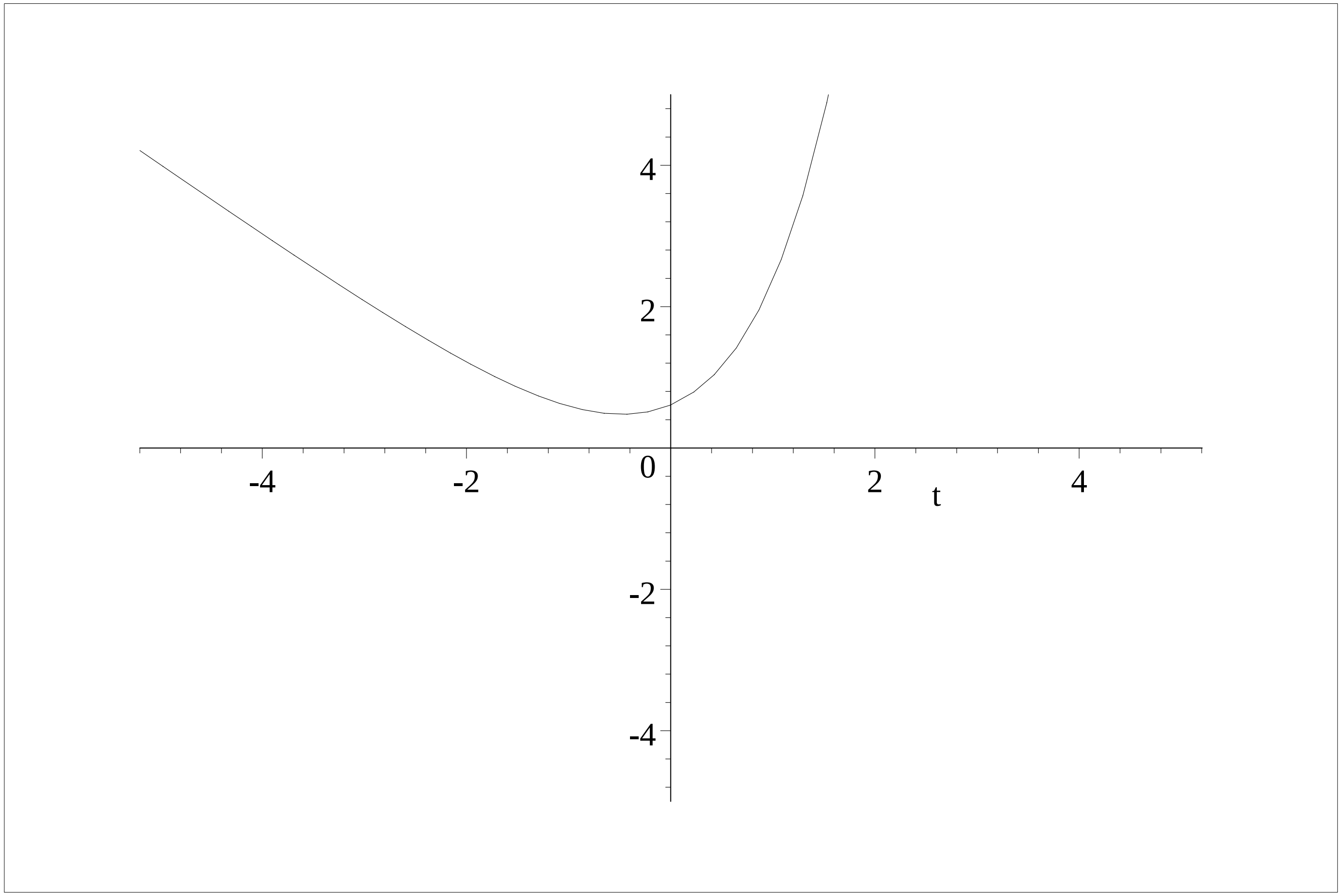
We shall soon learn the analytical techniques that will give us the solution : $y(t) = -t - 1 + e^t C$

Let us look at the solution curve, when $y(.4) = 1$

$\mathbf{1} = -.\mathbf{4} - \mathbf{1} + \mathbf{e}^{.4}\mathbf{C}$

$\mathbf{C} = \frac{2.4}{e^{.4}}$

$y = -t - 1 + \frac{2.4}{e^{.4}} e^t$



Approximating the solution curves by using the slope field

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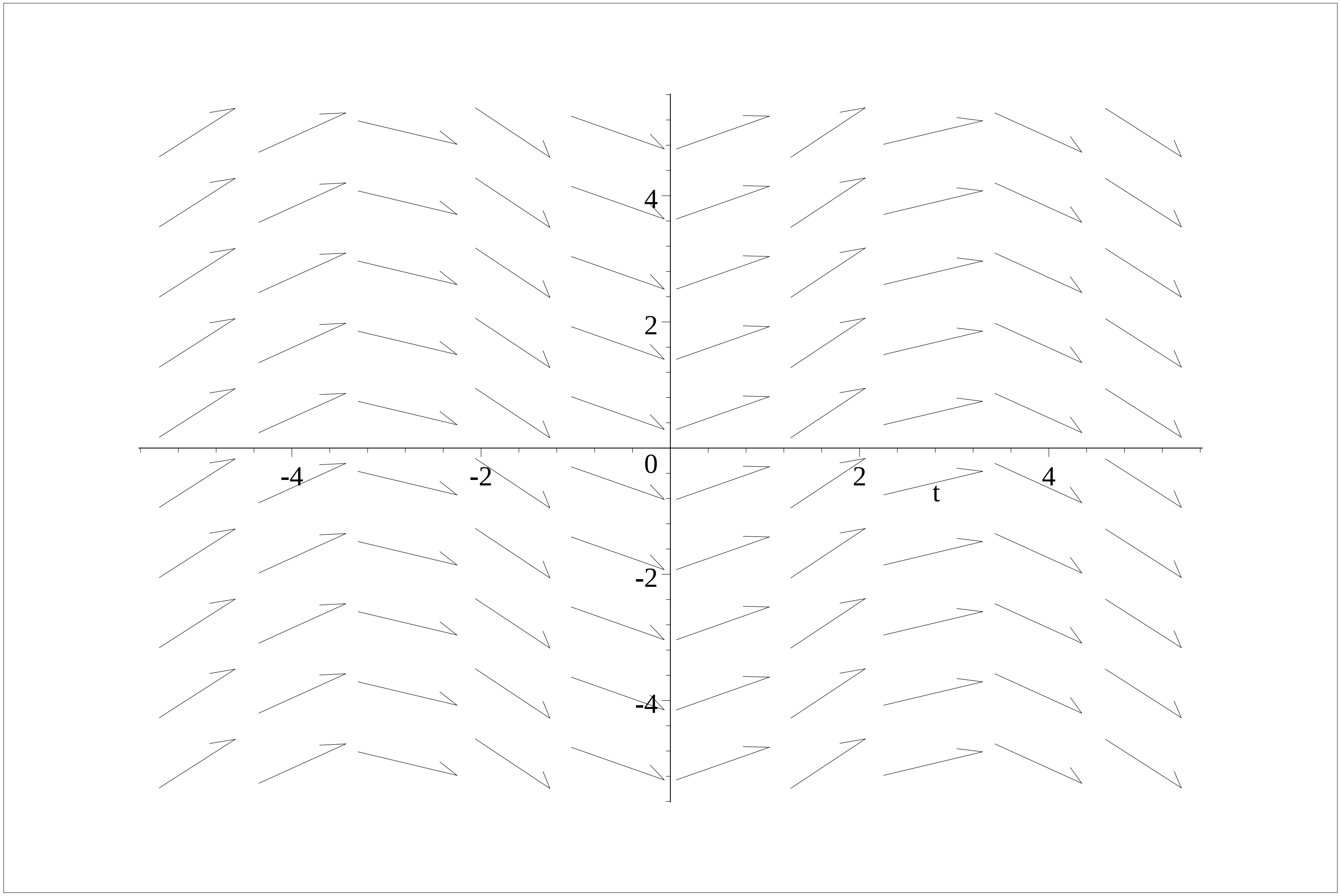
Slope fields of Differential Equations of the type: $\frac{dy}{dt} = f(t)$

$$\frac{dy}{dt} = \sin \mathbf{t}$$



look at the parallel segments as we go along the y-axis

The slope field is determined solely by the t –coordinates

on the other hand

For the differential equation of the type

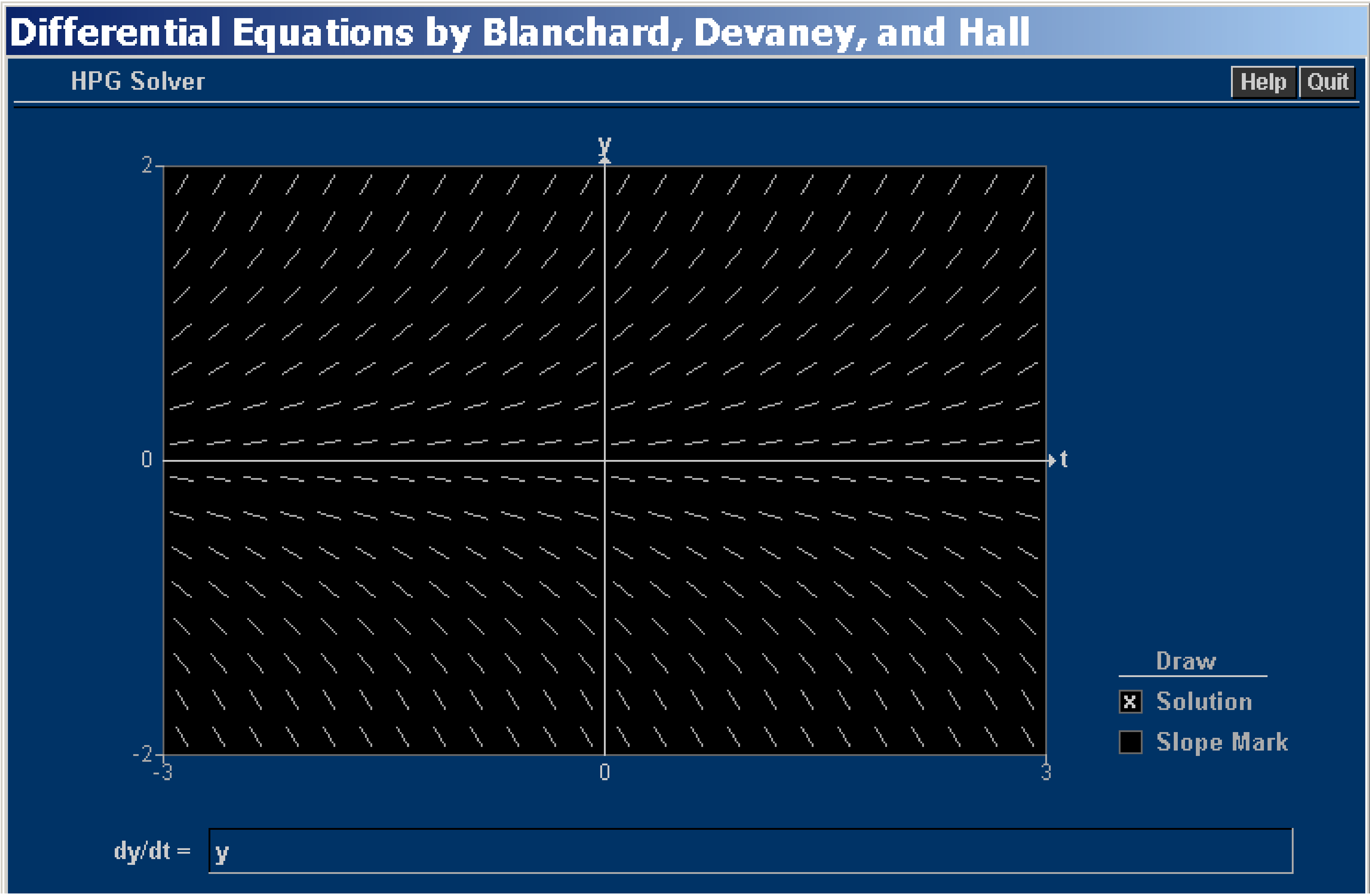
$\frac{dy}{dt} = f(y)$ **, where** $\frac{dy}{dt}$ **is a function of** y **only**

$\boxed{\text{An equation of the type } \frac{dy}{dt} = f(y) \text{ is called an autonomous equation}}$

Example:

Let us look at a slope field of

$\frac{dy}{dt} =$ **y**


Differential Equations by Blanchard, Devaney, and Hall
HPG Solver
Help
Quit
y
2
0
-2
-3
0
3
t
Draw
Solution
Slope Mark
dy/dt =
y

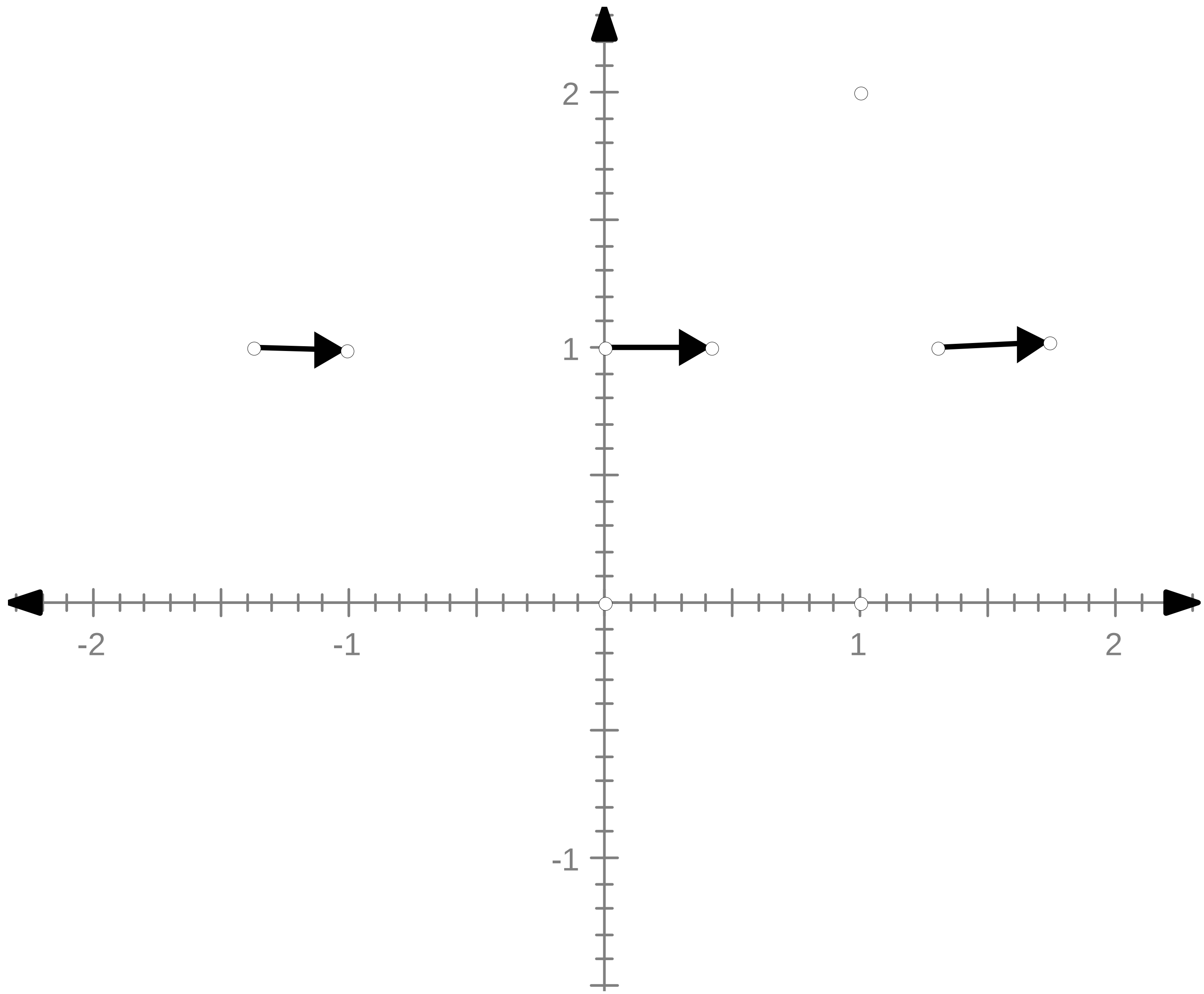
The slope field is completely determined by the y –coordinates
Please read the pages 36-47 in the text book

Some worked out Exercises from the Text Book:

#12 on the page 49

$\frac{dS}{dt} = S^3 - 2S^2 + S$

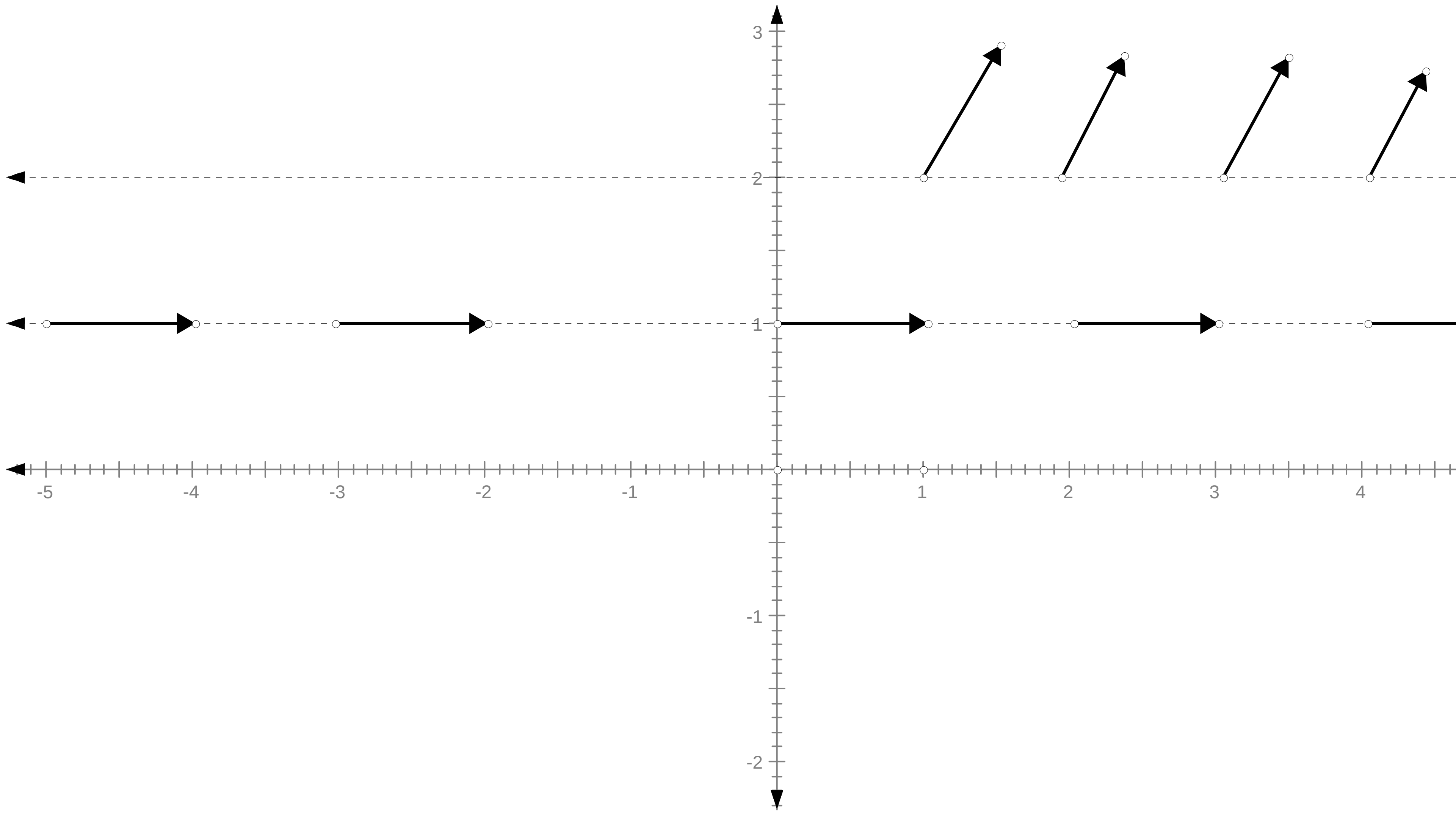
a)
Slope at $(0, 1)$
$t = 0, S = 1$
$1^3 - 2(1)^2 + 1 = 0$
Slope at (2,1) is still 0


2
1
-1
-2
-1
1
2

Slope at $(1,2)$ **is** $2^3 - 2(2)^2 + 2 = 2$

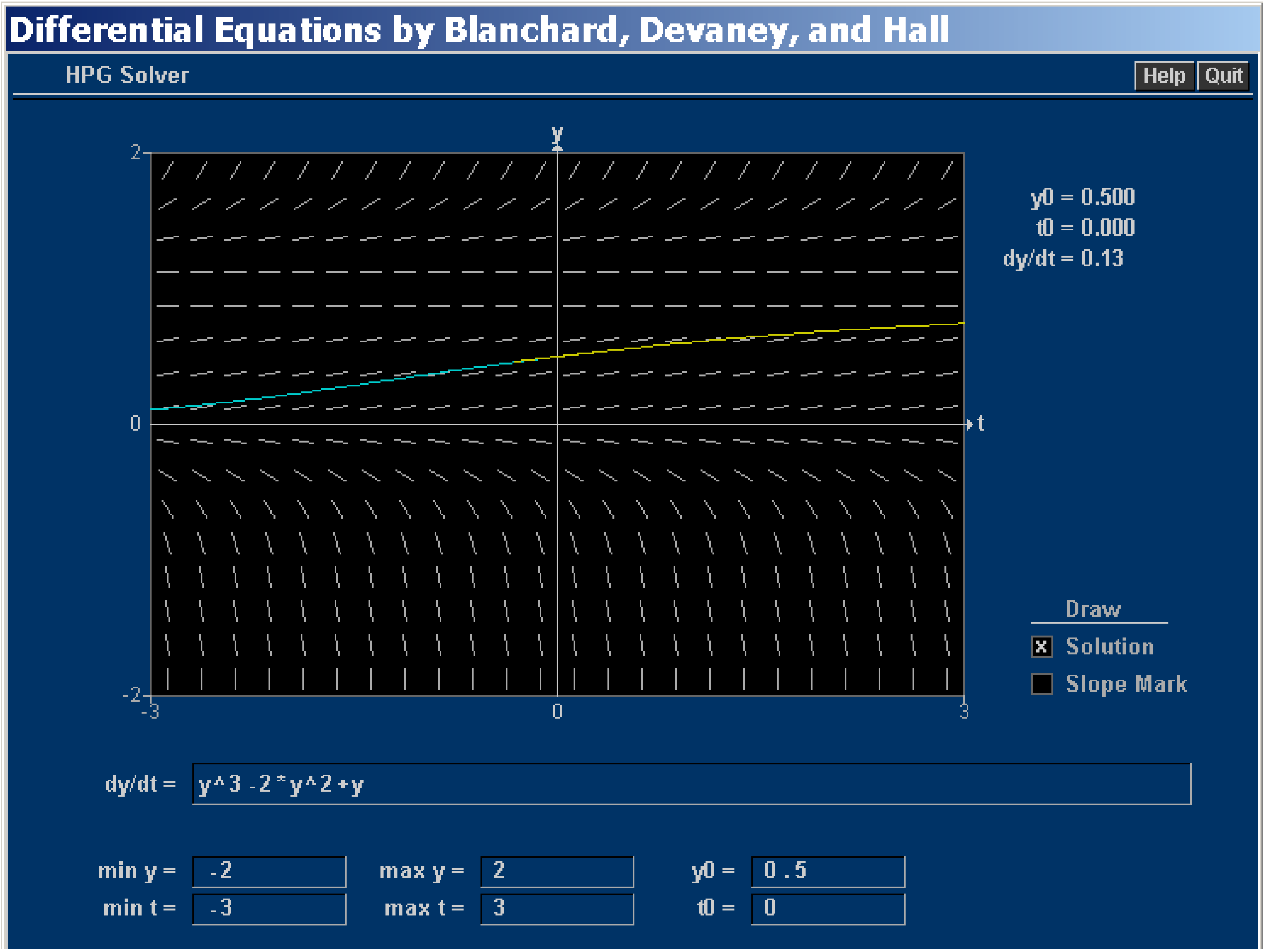
Slope at $(1,3)$ **is** $3^3 - 2(3)^2 + 3 = 12$

Since, this is an autonomous differential equation, the slope field is relatively easy to sketch

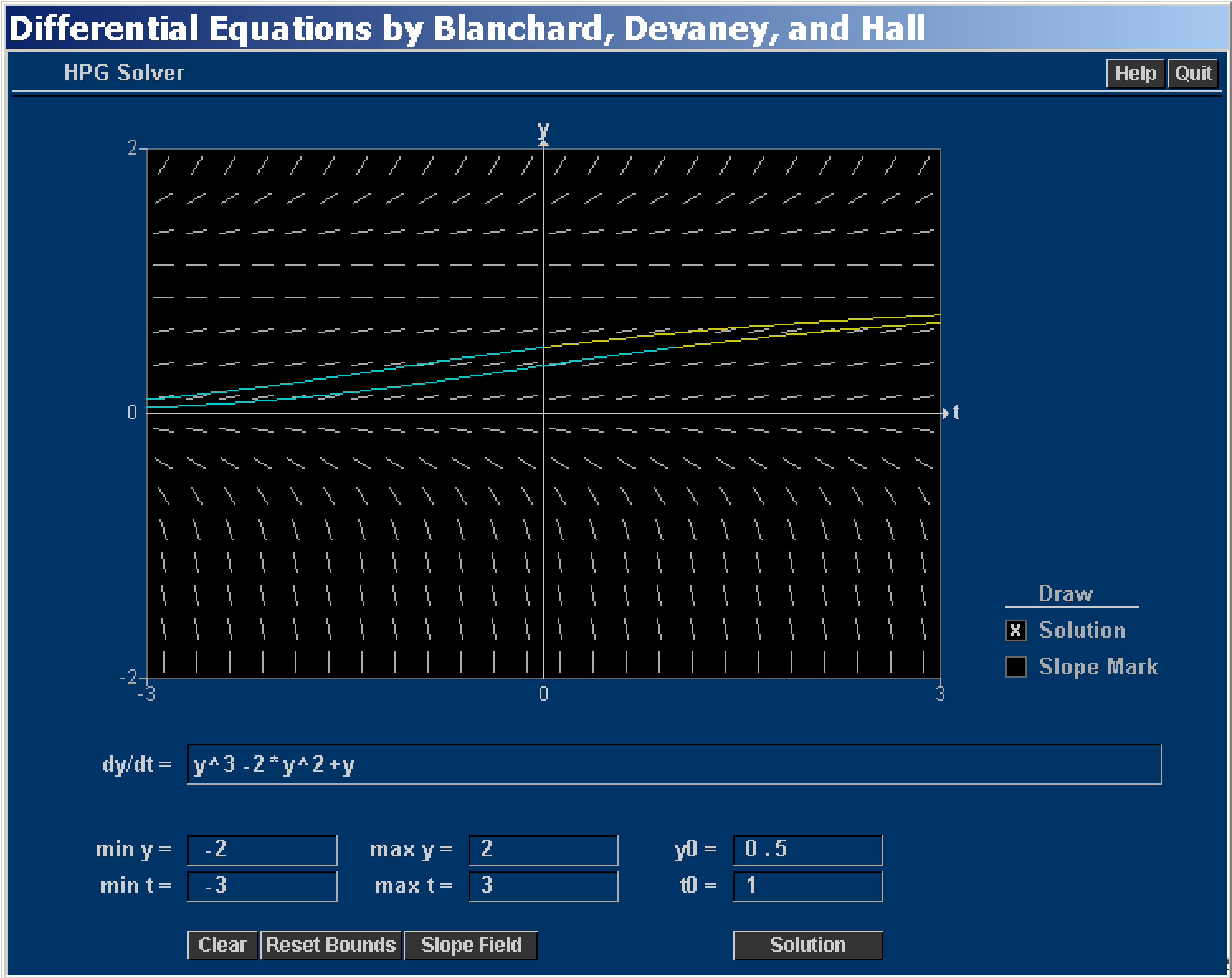


b)

$\mathbf{S}(0)=\frac{1}{2}$


Differential Equations by Blanchard, Devaney, and Hall
HPG Solver
Help
Quit
y
t
2
0
-2
-3
0
3
y0 = 0.500
t0 = 0.000
dy/dt = 0.13
Draw
Solution
Slope Mark
dy/dt = y^3 -2*y^2+y
min y = -2
max y = 2
y0 = 0.5
min t = -3
max t = 3
t0 = 0

$\mathbf{S}(1)=\frac{1}{2}$



Differential Equations by Blanchard, Devaney, and Hall
HPG Solver
Help
Quit
y
2
0
-2
t
-3
0
3
Draw
Solution
Slope Mark
dy/dt =
y^3 -2*y^2 +y
min y =
-2
max y =
2
y0 =
0.5
min t =
-3
max t =
3
t0 =
1
Clear
Reset Bounds
Slope Field
Solution

I am not showing the solutions sketched by hand, can do that on demand.

Page 51:

19.

$$\frac{dv_c}{dt} = \frac{V(t) - v_c}{RC}$$

Given that $V(t) = K$ **for all** t

changes to

$$\frac{dv_c}{dt} = \frac{K - v_c}{RC}$$

$$\frac{dv_c}{K - v_c} = \frac{dt}{RC}$$

$$\int \frac{dv_c}{K - v_c} = \int \frac{dt}{RC}$$

$$-\ln|K - v_c| = \frac{t}{RC} + \mathbf{A}$$
$$\ln|K - v_c| = -\frac{t}{RC} + \mathbf{A}$$
$$|K - v_c| = \mathbf{e}^{-\frac{t}{RC}}\mathbf{e}^{A}$$
$$|K - v_c| = \mathbf{d}\mathbf{e}^{-\frac{t}{RC}} \qquad \mathbf{d} = \mathbf{e}^{A}$$

We may let d **have either positive or negative values**

K – v$_c$= **de**$^{-\frac{t}{RC}}$
$v_c = K - de^{-\frac{t}{RC}}$ **is a general solution**

20.

Initial value $v_c(0) = 1$

$$\mathbf{V}(t) = \begin{cases} K & \text{for} \quad 0 \le t < 3 \\ 0 & \text{for} \quad t > 3 \end{cases}$$

Between 0 **and** 3

$v_c = K - de^{-\frac{t}{RC}}$ **applies**

given that $v_c(0) = 1$
1 $=$ **K – de**^{0}
d $=$ **K – 1**

$v_c = K - (K-1)e^{-\frac{t}{RC}}$ **for** $0 \le t < 3$

For $t > 3$

$\frac{dv_c}{dt} = \frac{0 - v_c}{RC}$

or

$\frac{dv_c}{dt} = -\frac{v_c}{RC}$

or

$\frac{dv_c}{v_c} = -\frac{dt}{RC}$

or

$\ln|v_c| = -\frac{1}{RC}\mathbf{t} + \mathbf{A}$

or

$\mathbf{v}_c = \mathbf{ae}^{\left(-\frac{1}{RC}t\right)}$

If we assume that $v_c(t)$ **is a continuous function of** t

then we may let the equation for $0 \leq t < 3$

determine the value for v_c **at** $t = 3$

taking the equation

$\mathbf{v}_c = \mathbf{K} - (K-1)\mathbf{e}^{-\frac{t}{RC}}$

We find that

$\mathbf{v}_c(3) = \mathbf{K} - (K-1)\mathbf{e}^{-\frac{3}{RC}}$

Combining this with

$\mathbf{v}_c(t) = \mathbf{a}\mathbf{e}^{\left(-\frac{1}{RC}t\right)}$

$\mathbf{v}_c(3) = \mathbf{a}\mathbf{e}^{\left(-\frac{3}{RC}\right)}$

We get

$\mathbf{a}\mathbf{e}^{\left(-\frac{3}{RC}\right)} = \mathbf{K} - (K-1)\mathbf{e}^{-\frac{3}{RC}}$

Multiplication by $e^{\frac{3}{RC}}$ **gives**

$\mathbf{a} = \mathbf{K}\mathbf{e}^{\frac{3}{RC}} - (K-1)$

Therefore

for $t \geq 3$

$\mathbf{v}_c(t) = \mathbf{a}\mathbf{e}^{\left(-\frac{t}{RC}\right)}$

or

$v_c(t) = \left(Ke^{\frac{3}{RC}} - (K-1)\right)e^{\left(-\frac{t}{RC}\right)}$ **for** $t \geq 3$

Read the Euler's Method on the pages 55-56

To make the slope fields

for

$\frac{dy}{dt} = \mathbf{f}(\mathbf{y}, \mathbf{t})$

in a way that

Let us look at a demonstration of the procedure outlined on the page 56

#2 on the page 63

$\frac{dy}{dt} = t - y^2 = f(t,y)$, $y(0) = 1$, $0 \le t \le 1$, $\Delta t = 0.25$

$y(0) = 1$
$t_0 = 0$
$\mathbf{y}_0 = \mathbf{1}$

$\mathbf{t}_1 = .\mathbf{25}$
$\mathbf{y}_1 = \mathbf{y}_0 + \mathbf{f}(x_0, y_0)\Delta\mathbf{t}$
To approximate y for .25

at $(0,1)$
$\frac{dy}{dt} = \mathbf{0} - \mathbf{1}^2 = -\mathbf{1}$

therefore

for t=0 to t=.25

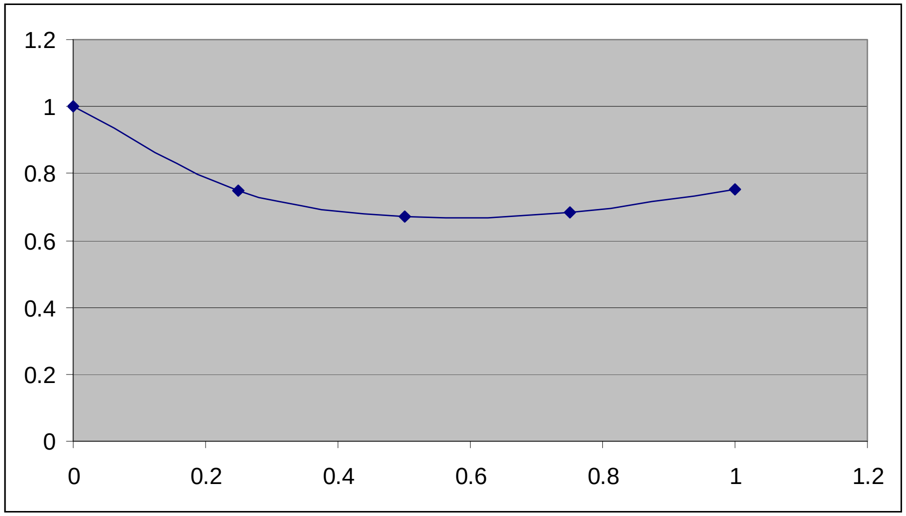
$\mathbf{y}_1 = \mathbf{1} + \mathbf{f}(0,1)(.25)$
$\mathbf{y}_1 = \mathbf{1} + (-1)(.25)$
$\mathbf{y}_1 = .\mathbf{75}$

$\mathbf{t}_2 = .\mathbf{5}$
$\mathbf{y}_2 = \mathbf{y}_1 + \mathbf{f}(t_1, y_1)(.25)$
$\mathbf{y}_2 = .\mathbf{75} + (.25 - .75^2)(.25) = \mathbf{0}.\mathbf{671875}$

$\mathbf{t}_3 = .\mathbf{75}$
$\mathbf{y}_3 = \mathbf{y}_2 + \mathbf{f}(t_2, y_2)(.25)$
$\mathbf{y}_3 = \mathbf{0}.\mathbf{671875} + (.5 - 0.671875^2)(.25) = \mathbf{0}.\mathbf{6840209961}$

$\mathbf{t}_4 = \mathbf{1}$
$\mathbf{y}_4 = \mathbf{y}_3 + \mathbf{f}(t_3, y_3)(.25)$
$\mathbf{y}_4 = \mathbf{0}.\mathbf{6840209961} + (.75 - 0.6840209961^2)(.25) = \mathbf{0}.\mathbf{7545498153}$

t	y
0	1
0.25	0.75
0.5	0.671875
0.75	0.684020996 1
1	0.754549815 3



Existence and uniqueness of an initial value problem

Given that

$\frac{dy}{dt} = f(t,y)$ **initial condition** $y(t_0) = y_0$

If $f(t,y)$ **is continuous in an open rectangle containing** (t_0, y_0)
then a solution exists near t_0 $(t_0 - \epsilon, t_0 + \epsilon)$ **(EXISTENCE Theorem)**

Moreover, if both $f(t,y)$ **and** $\frac{\partial f}{\partial y}$ **are continuous in an open rectangle containining** (t_0, y_0)**, then we have a unique solution near** t_0 **(UNIQUENESS Theorem)**

 # 14 on the page 75

$\frac{dy}{dt} = y^3$**,** $y(0) = 1$

$\mathbf{f}(\mathbf{t}, \mathbf{y}) = \mathbf{y}^3$

$\frac{\partial f}{\partial y} = \mathbf{3y}^2$

are both continuous

a)

$\frac{dy}{dt} = \mathbf{y}^3$

$\to$

$\frac{dy}{y^3} = \mathbf{dt}$

$-\frac{1}{2y^2} = t + C$ **apply the initial condition** $y(0) = 1$

$-\frac{1}{2(1)^2} = \mathbf{0} + \mathbf{C}$

$\mathbf{C} = -\frac{1}{2}$

$-\frac{1}{2y^2} = \mathbf{t} - \frac{1}{2}$

$-\frac{1}{2y^2} = \frac{2t-1}{2}$

$-\frac{1}{y^2} = \mathbf{2t - 1}$

$\mathbf{y}^2 = -\frac{1}{2t-1}$

$\mathbf{y}^2 = \frac{1}{1-2t}$

$\mathbf{y} = \pm\sqrt{\frac{1}{1-2t}}$

$\mathbf{y}(0) = \mathbf{1}$

therefore

$\mathbf{y}(t) = \sqrt{\frac{1}{1-2t}}$

is our solution

Since we must have

$\mathbf{1 - 2t} > \mathbf{0}$

1 $>$ **2t**
t $<\frac{1}{2}$

the domain is $\left(-\infty, \frac{1}{2}\right)$

as $t \to \frac{1}{2}^{-}$**,** $y \to \infty$
as $t \to -\infty$**,** $y \to 0$

Please work on the sections 1.3, 1.4 and 1.5

In case, you would like more explanations, please let me know,
I shall write up more.