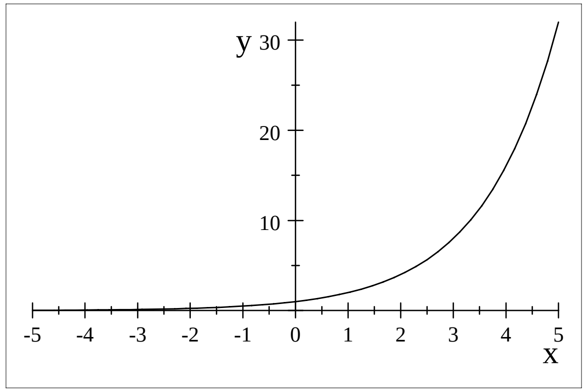
$f(x) = 2^x$



The above is an example of an expenential function,

in general an exponential function looks like

$f(x) = a^x$, **where** $a > 0, a \neq 1$

If we invest an amount P **in an account that pays interest at the rate of** r (**proportion**) **compounded** m **times a year then the final amount** A **after** t **years is**

$$\mathbf{A} = \mathbf{P}\left(1 + \frac{r}{m}\right)^{mt}$$

Example,

$**10000 are invested in an account that pays 9**% **interest**. **What will be the final amount after 15 years if**
a) **the interest is compounded annually**
$A = 10000(1 + .09)^{15} = \$36424.82$
b) **the interest is compounded semiannually**

$$A = 10000\left(1 + \frac{.09}{2}\right)^{2\times 15} = \$37453.18$$

c) the interest is compounded quarterly

$$A = 10000\left(1 + \frac{.09}{4}\right)^{4\times 15} = \$38001.35$$

d) the interest is compounded monthly

$$A = 10000\left(1 + \frac{.09}{12}\right)^{12\times 15} = \$38380.43$$

e) the interest is compounded daily (ingnore the leap years)

$$\mathbf{A = 10000}\left(1 + \frac{.09}{365}\right)^{365\times 15} = \$\mathbf{38567.84}$$

f) the interest is compounded hourly

$$\mathbf{10000}\left(1 + \frac{.09}{365 \times 24}\right)^{24\times 365\times 15} = \$\mathbf{38573.99}$$

h) the interest is compounded each second

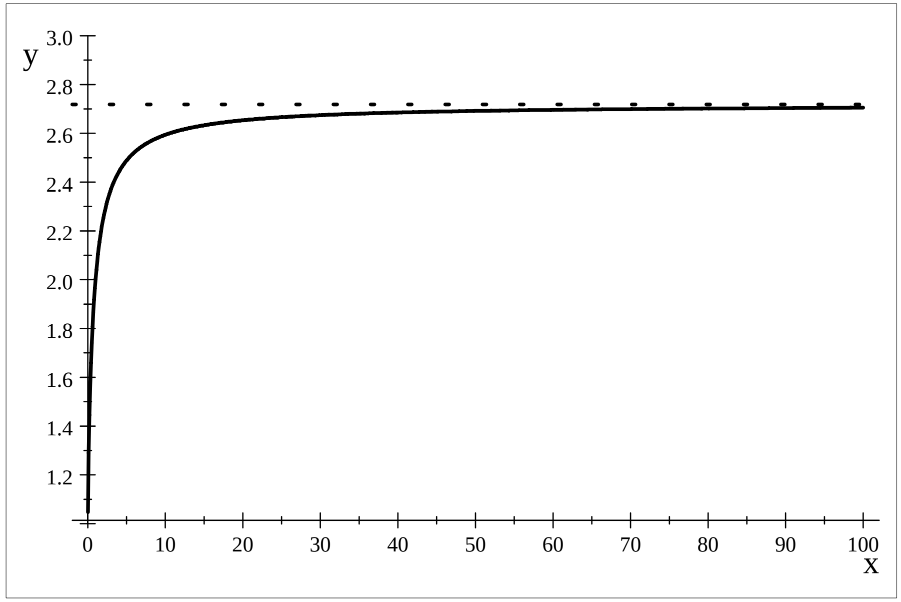
$$10000\left(1 + \frac{.09}{365 \times 24 \times 60 \times 60}\right)^{365\times 24\times 60\times 60\times 15} = \$38574.26$$

....

Let us look at the growth of

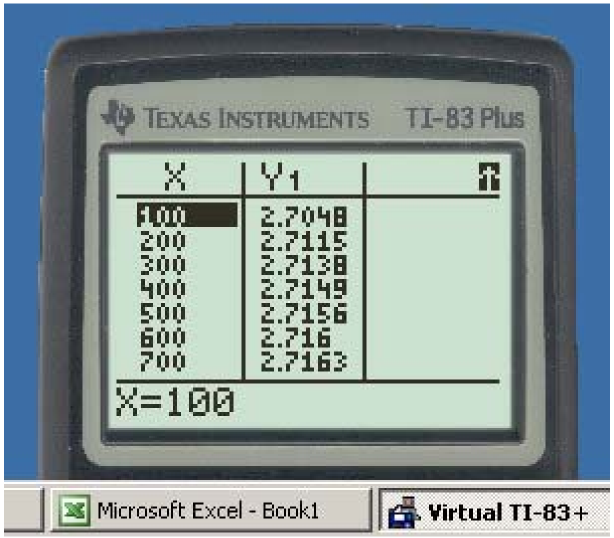
$y = \left(1 + \frac{1}{x}\right)^x$ **as** $x \to \infty$

$$y = \left(1 + \frac{1}{x}\right)^x$$



$$\mathbf{g}(x) = \left(1 + \frac{1}{x}\right)^x$$

$$\mathbf{g}\begin{pmatrix} 10 \\ 100 \\ 200 \\ 1000 \end{pmatrix} = \begin{pmatrix} 2.59374246 \\ 2.704813829 \\ 2.711517123 \\ 2.716923932 \end{pmatrix}$$



In fact

$$\mathbf{Lim}_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=\mathbf{e}$$

$\mathbf{e}\cong\mathbf{2.718\,281\,828}$

This transcendetal number e is a good way to model contious growth

$f(x)=e^{x}$ **is called the natural exponential function**

#46 on the page 302

$$\mathbf{500}\left(1+\frac{.18}{12}\right)^{12\times 1}=\$\mathbf{597.81}$$

Note that an exponential function is a 1-1 function and therefore has inverse

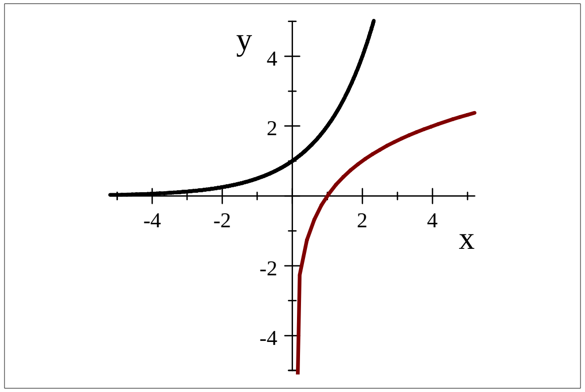
$\mathbf{2}^{u} = \mathbf{2}^{v} \Rightarrow \mathbf{u} = \mathbf{v}$

....

$\mathbf{f}(x) = \mathbf{2}^{x}$

x	f(x)
-5	$2^{-5} = \frac{1}{32}$
-4	$\frac{1}{16}$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8
4	16
5	32

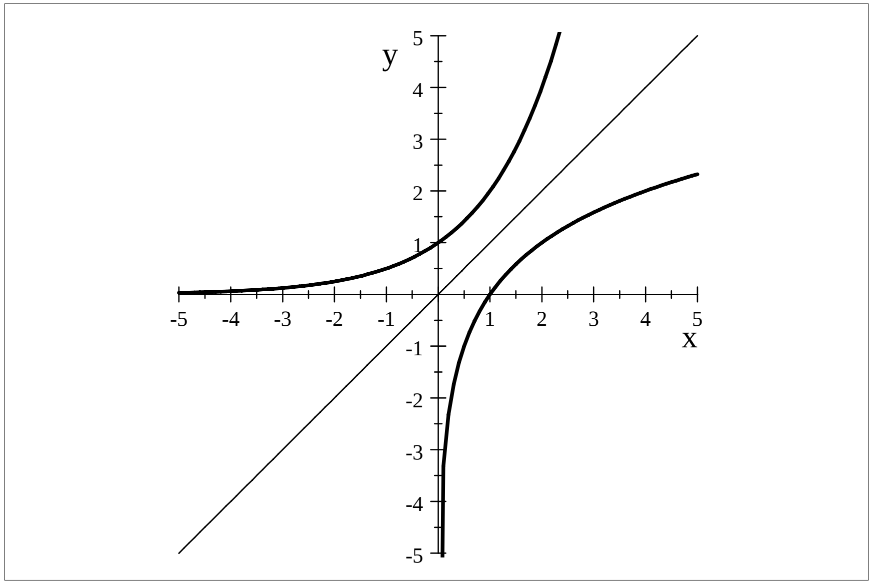
2^{x}



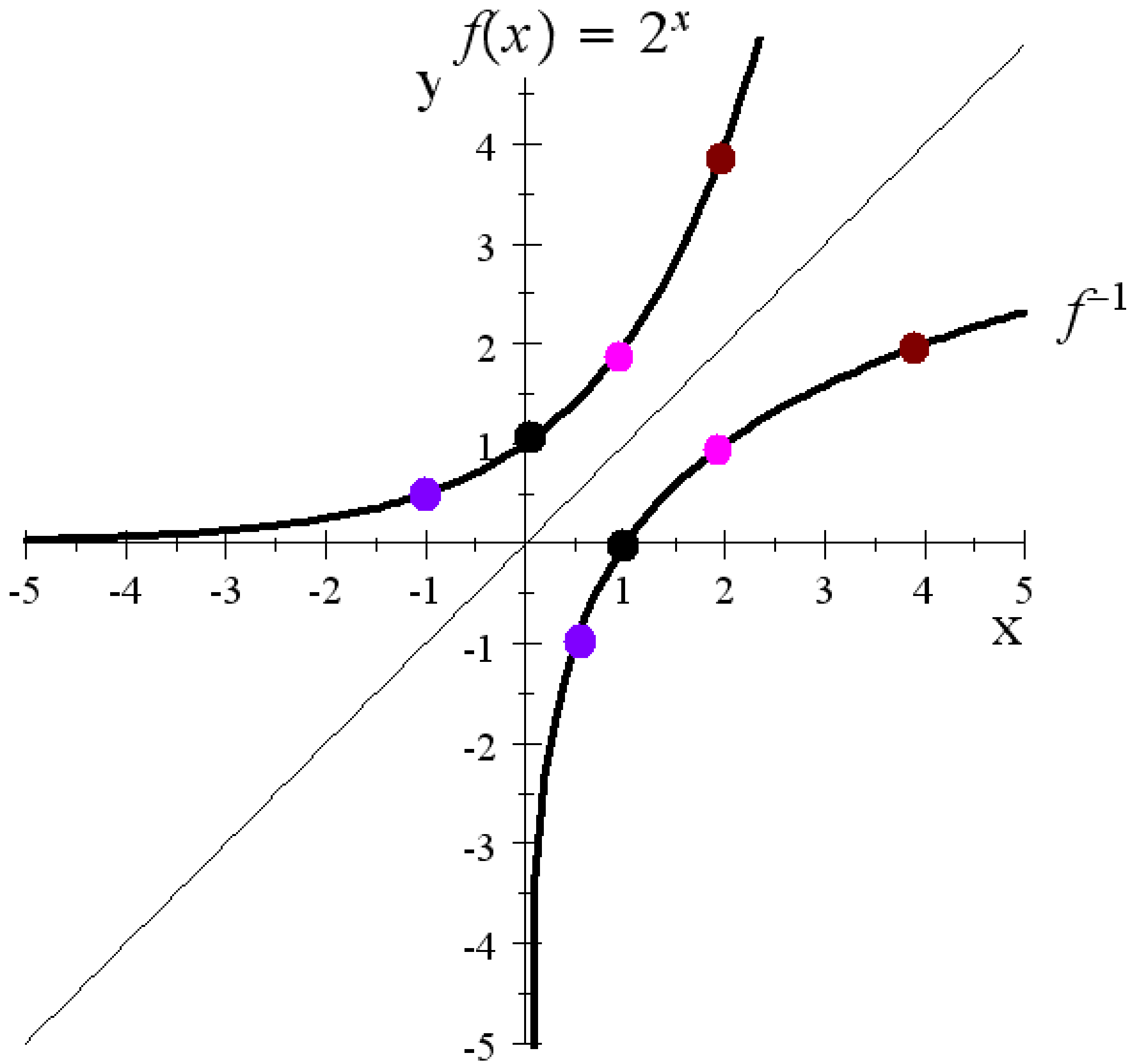
Work on the section 4.2 please

Noted that

$f(x) = 2^x$



f^{-1} **exists and its graph is the reflection of the graph of** f **by the line** $y = x$



For

$f(x) = 2^x$ **we say that** $f^{-1}(x) = \log_2 x$

$2^a = b \quad \Leftrightarrow \quad \log_2 b = a$

$2^4 = 16 \quad \log_2 16 = 4 \quad \log_2 2^4 = 4$

$g(x) = 10^x \quad \Leftrightarrow \quad g^{-1}(x) = \log_{10} x$

$\mathbf{10}^1 = \mathbf{10} \quad \Leftrightarrow \quad \log_{10}\mathbf{10} = \mathbf{1}$

$10^2 = 100 \quad \Leftrightarrow \quad \log_{10} 10^2 = 2$

Question:

What is $\log_5 125$

$\log_5 \mathbf{125} = \log_5 \mathbf{5}^3 = \mathbf{3}$

What is $\log_3 81$

$3^4 = 81$

$\log_3 \mathbf{3}^4 = \mathbf{4}$

What is $\log_9 81$

$\log_9 \mathbf{9}^2 = \mathbf{2}$

Remember that

For $a > 0$

$$\mathbf{a}^u \mathbf{a}^v = \mathbf{a}^{u+v}$$
$$\frac{a^u}{a^v} = \mathbf{a}^{u-v}$$
$$(a^u)^v = \mathbf{a}^{uv}$$

For $a > 0 \qquad a \neq 1 \qquad M > 0, N > 0$

$$\log_a (MN) = \log_a \mathbf{M} + \log_a \mathbf{N}$$
$$\log_a \left(\frac{M}{N}\right) = \log_a \mathbf{M} - \log_a \mathbf{N}$$
$$\log_a \mathbf{M}^p = \mathbf{p} \log_a \mathbf{M}$$

In particular

$$\log_a \mathbf{a} = \mathbf{1}$$
$$\log_a \mathbf{1} = \mathbf{0}$$

Two special logarthmic functions are

$y = \log_e x$ **also denoted by** $y = \ln x$ y **is natural log of** x

$y = \log_{10} x$ **also denoted by** $y = \log x$ y **is common log of** x

$\log_{10} \mathbf{10} = \mathbf{1}$
$\log_{10} \mathbf{100} = \mathbf{2}$

$\log_{10} 50$ **is somewhere between 1 and 2**

$\log_{10} \mathbf{50} \cong \mathbf{1.\,698\,970\,004\,336\,018\,804\,786\,261\,105\,275\,506\,973\,231\,810\,118\,537\,891\,458\,689\,57}$

$\ln 1 = 0$

$\ln \mathbf{e} = \log_e \mathbf{e} = \mathbf{1}$
$e \cong 2.\,718\,281\,828\,459\,045\,235\,360\,287\,471\,352\,662\,497\,757\,247\,093\,699\,959\,574\,966\,97$

$\ln \mathbf{e}^2 = \mathbf{2}$

$e^2 \cong 7.\,389\,056\,098\,930\,650\,227\,230\,427\,460\,575\,007\,813\,180\,315\,570\,551\,847\,324\,087\,13$

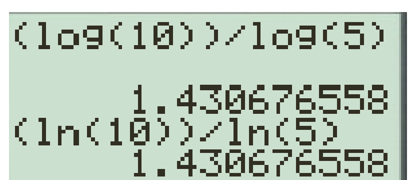
$\ln \mathbf{2} \cong \mathbf{0.\,693\,147\,180\,559\,945\,309\,417\,232\,121\,458\,176\,568\,075\,500\,134\,360\,255\,254\,120\,68}$

$\log_5 \mathbf{5} = \mathbf{1}$

$\log_5 \mathbf{25} = \mathbf{2}$

$\log_5 \mathbf{10} \cong \mathbf{1.\,430\,676\,558\,073\,393\,050\,670\,106\,568\,763\,965\,632\,069\,791\,932\,079\,760\,449\,321\,98}$

$$\log_a M = \frac{\ln M}{\ln a} \qquad \log_a M = \frac{\log M}{\log a}$$



Question?

$\log_2 a = b \quad \Leftrightarrow \quad 2^b = a$

Solve the equations

1.

$\log_2 \mathbf{x} = \mathbf{3}$

$\mathbf{2}^3 = \mathbf{x}$

$\mathbf{x} = \mathbf{8}$

$\log_2 \mathbf{a} = \mathbf{b} \quad \Leftrightarrow \quad \mathbf{2}^b = \mathbf{a}$

2.

Solve $\log_2 x(x-2) = 3$

$2^3 = x(x-2)$

$\mathbf{x}(x-2) = \mathbf{2}^3$

$\mathbf{x}^2 \mathbf{-2x} = \mathbf{8}$

$\mathbf{x}^2 \mathbf{-2x-8} = \mathbf{0}$

$(x-4)(x+2) = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{-2}, \mathbf{x} = \mathbf{4}$

Substitute back in

Solve $\log_2 x(x-2) = 3$

$\log_2(-2)(-2-2) = 3$ **is true**
$\log_2(4)(4-2) = 3$

$\log_2(8) = 3$ **Correct**

We can now start working on the Chapter 4 exercises

More Examples:

Examples:
1.
$2^4 = 16$ **means** $\log_2 16 = 4$

2.
$5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

means

$\log_5\left(\frac{1}{25}\right) = \mathbf{-2}$

3.
Note that

x	$y = 2^x$	$\log_2 y$
-3	$2^{-3} = \frac{1}{8}$	$\log_2\left(\frac{1}{8}\right) = -3$
-2	$2^{-2} = \frac{1}{4}$	$\log_2\left(\frac{1}{4}\right) = -2$
-1	$2^{-1} = \frac{1}{2}$	$\log_2\left(\frac{1}{2}\right) = -1$
0	$2^0 = 1$	$\log_2(1) = 0$
1	$2^1 = 1$	$\log_2(2) = 1$
2	$2^2 = 4$	$\log_2(4) = 2$
3	$2^3 = 8$	$\log_2(8) = 3$

Example 4:

$\log_5(625) = \mathbf{4}$

Recall that for $a > 0$ **and** $a \neq 1$

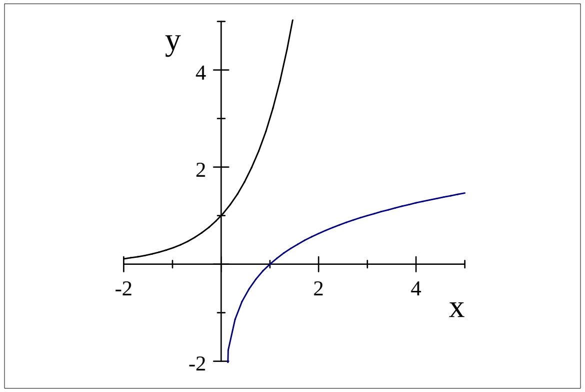
The domain of the function given by $y = a^x$ **is** $(-\infty, \infty)$ **and the range is** $(0, \infty)$

Therefore the domain of the function given by $y = \log_a x$ **is** $(0, \infty)$ **and the range is** $(-\infty, \infty)$

For example, **in the picture below**,

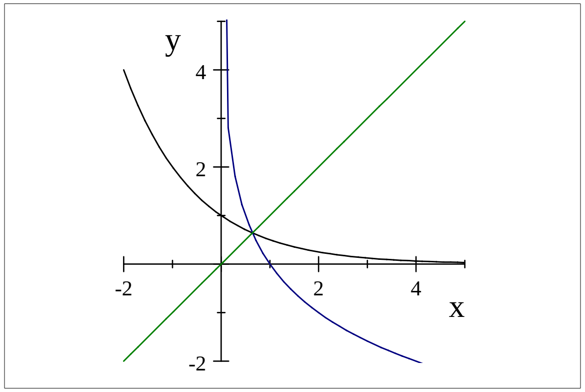
black graph is that of $y = 3^x$ **and the blue graph is that of** $y = \log_3 x$

3^x



In the following picture, **the black graph is that of** $y = \left(\frac{1}{2}\right)^x$ **and the blue graph is that of** $y = \log_{(1/2)}(x)$. **The green one is a graph of** $y = x$

2^{-x}



....

Note again that

If $a > 0$ **and** $a \neq 1$

and $M > 0 \qquad N > 0$

$\log_a \mathbf{1} = \mathbf{0}$

$\log_a \mathbf{a} = \mathbf{1}$

$\log_a (MN) = \log_a \mathbf{M} + \log_a \mathbf{N}$

$\log_a \left(\frac{M}{N}\right) = \log_a \mathbf{M} - \log_a \mathbf{N}$

$\log_a \mathbf{M}^p = \mathbf{p} \log_a \mathbf{M}$

Examples of the usage of the properties of Logarithms:

To write $\log \frac{\sqrt{abc}}{b^3 c^4}$ **in terms of** $\log a, \log b, \log c$

recall

I: $\log MN = \log M + \log N$

II: $\log\left(\frac{M}{N}\right) = \log M - \log N$

III: $\log M^p = p\log M$

One way

$\log \frac{\sqrt{abc}}{b^3c^4}$

$= \log\left(\sqrt{abc}\right) - \log\left(b^3c^4\right)$ **applied the rule II**

$=\log(abc)^{1/2} - \log\left(b^3c^4\right)$ **rewrote it**

$=\frac{1}{2}\log(abc) - \log\left(b^3c^4\right)$ **used III on the first term**

$=\frac{1}{2}(\log a + \log b + \log c) - \left(\log b^3 + \log c^4\right)$ **used I on the second term**

$=\frac{1}{2}\log a + \frac{1}{2}\log b + \frac{1}{2}\log c - \log b^3 - \log c^4$ **expanded**

$=\frac{1}{2}\log a + \frac{1}{2}\log b + \frac{1}{2}\log c - 3\log b - 4\log c$ **used III**

$=\frac{1}{2}\log a + \frac{1}{2}\log b - 3\log b + \frac{1}{2}\log c - 4\log c$

$=\frac{1}{2}\log a - \frac{5}{2}\log b - \frac{7}{2}\log c$

....

More examples of solving equations involving logs

I: $\log MN = \log M + \log N$

II: $\log\left(\frac{M}{N}\right) = \log M - \log N$

III: $\log M^p = p\log M$

Example 1:

Solve

$\log(5x-4) = \log 3 + \log(x-1)$

$\log(5x-4) = \log 3(x-1)$ **using I**

$(5x-4) = 3(x-1)$ **because** $\log$ **is 1-1**

$\mathbf{5x - 4 = 3x - 3}$

$\mathbf{5x - 3x = -3 + 4}$

$\mathbf{2x = 1}$

$\mathbf{x} = \frac{1}{2}$

another example

solve

$\log(2x-3)=\log \mathbf{2} + \log(x-1)$

$\log(2x-3)=\log \mathbf{2}(x-1)$

$\mathbf{2x - 3 = 2}(x-1)$

$\mathbf{2x - 3 = 2x - 2}$

$\mathbf{2x - 2x = -2 + 3}$

$0 = 1$ **BAD STATEMENT**

No solution

Example 2:

Solve

$\log(3x-7)=\log \mathbf{1} + \log(x-2)$

$\log(3x-7) = \log 1(x-2)$

$\log(3x-7)=\log(x-2)$

$3x - 7 = x - 2$
$3x - x = 7 - 2$
$2x = 5$
$\frac{2x}{2} = \frac{5}{2}$
$x = \frac{5}{2}$

or could note $\log 1 = 0$ **and do it with shorter # of steps**.

Check:
$\log\left(3\left(\frac{5}{2}\right) - 7\right) = \log 1 + \log\left(\frac{5}{2} - 2\right)$

Example 3:

Solve

$25^{3x+1} = 8^{2x}$

$5^{2(3x+1)} = 2^{3(2x)}$
$\ln 5^{2(3x+1)} = \ln 2^{3(2x)}$
$2(3x + 1)\ln 5 = 6x \ln 2$
$6x \ln 5 + 2\ln 5 = 6x \ln 2$
$6x \ln 5 - 6x \ln 2 = -2\ln 5$
$(6\ln 5 - 6\ln 2)x = -2\ln 5$
$x = -\frac{2\ln 5}{6\ln 5 - 6\ln 2}$

To solve

$\log_3(x-1)+\log_3(x+2)= \mathbf{2}$

$\log_3(x-1)(x+2)= \mathbf{2}$

$$\boxed{\log_3 M + \log_3 N = \log_3(MN)}$$

$(x-1)(x+2)= \mathbf{3}^2$

$\mathbf{x^2 - x - 2 = 9}$

$\mathbf{x^2 - x - 2 - 9 = 0}$

$\mathbf{x^2 - x - 11 = 0}$

$$\mathbf{x} = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-11)}}{2 \cdot 1}$$

$$\mathbf{x} = \frac{1 \pm \sqrt{1+44}}{2}$$

$$x = \frac{1 \pm \sqrt{45}}{2}$$

Proposed solutions are

$$\frac{1-\sqrt{45}}{2}, \frac{1+\sqrt{45}}{2}$$

try these back in the original equation

$\log_3\left(\frac{1-\sqrt{45}}{2} - 1\right) + \log_3\left(\frac{1-\sqrt{45}}{2} + 2\right) = 2$ **is false**, $\left(\frac{1-\sqrt{45}}{2} - 1\right)$ **is not even in the domain of** $\log_3 t$

$$\log_3\left(\frac{1+\sqrt{45}}{2}-1\right)+\log_3\left(\frac{1+\sqrt{45}}{2}+2\right) = 2$$

$$\log_3\left(\frac{1+\sqrt{45}}{2}-1\right)\left(\frac{1+\sqrt{45}}{2}+2\right)= \mathbf{2}$$

Note that

$$\left(\frac{1+\sqrt{45}}{2}-1\right)\left(\frac{1+\sqrt{45}}{2}+2\right)= \mathbf{3}\sqrt{5}+\mathbf{10}$$

$$\log_3\left(\frac{1+\sqrt{45}}{2}-1\right)\left(\frac{1+\sqrt{45}}{2}+2\right)$$

$$=\log_3\left(3\sqrt{5}+10\right)$$

$$=\log_3\left(3\sqrt{5}+10\right)$$

$$=\log_3\left(3\sqrt{5}+2\sqrt{5}\sqrt{5}\right)$$

$$=\log_3\left(\left(3+2\sqrt{5}\right)\sqrt{5}\right)$$

$$\log_3\left(\left(3+2\sqrt{5}\right)\sqrt{5}\right) = 2$$

will mean

$$\left(3+2\sqrt{5}\right)\sqrt{5}= \mathbf{3}^2$$

$\left(3+2\sqrt{5}\right)\sqrt{5} = 9$ **is FALSE**

No solution

Example:

Find the inverse of the function

$\mathbf{f}(x) = \mathbf{5 \cdot 3^{x-2}}$

$\mathbf{y = 5 \cdot 3^{x-2}}$

$\mathbf{x \Leftrightarrow y}$

$\mathbf{x = 5 \cdot 3^{y-2}}$

To solve for y

$\mathbf{x = 5 \cdot 3^{y-2}}$

$\mathbf{5 \cdot 3^{y} = x + 2}$

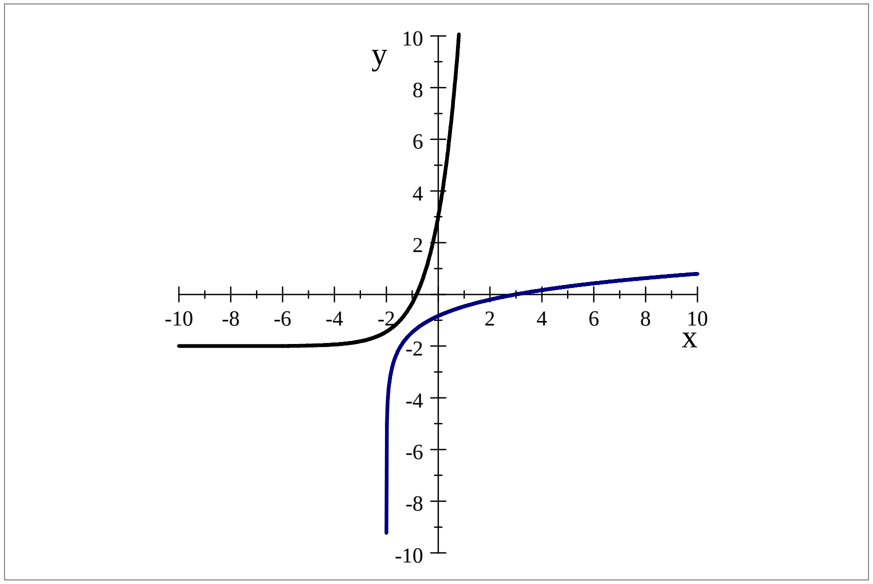
$$\mathbf{3}^{y} = \frac{x+2}{5}$$

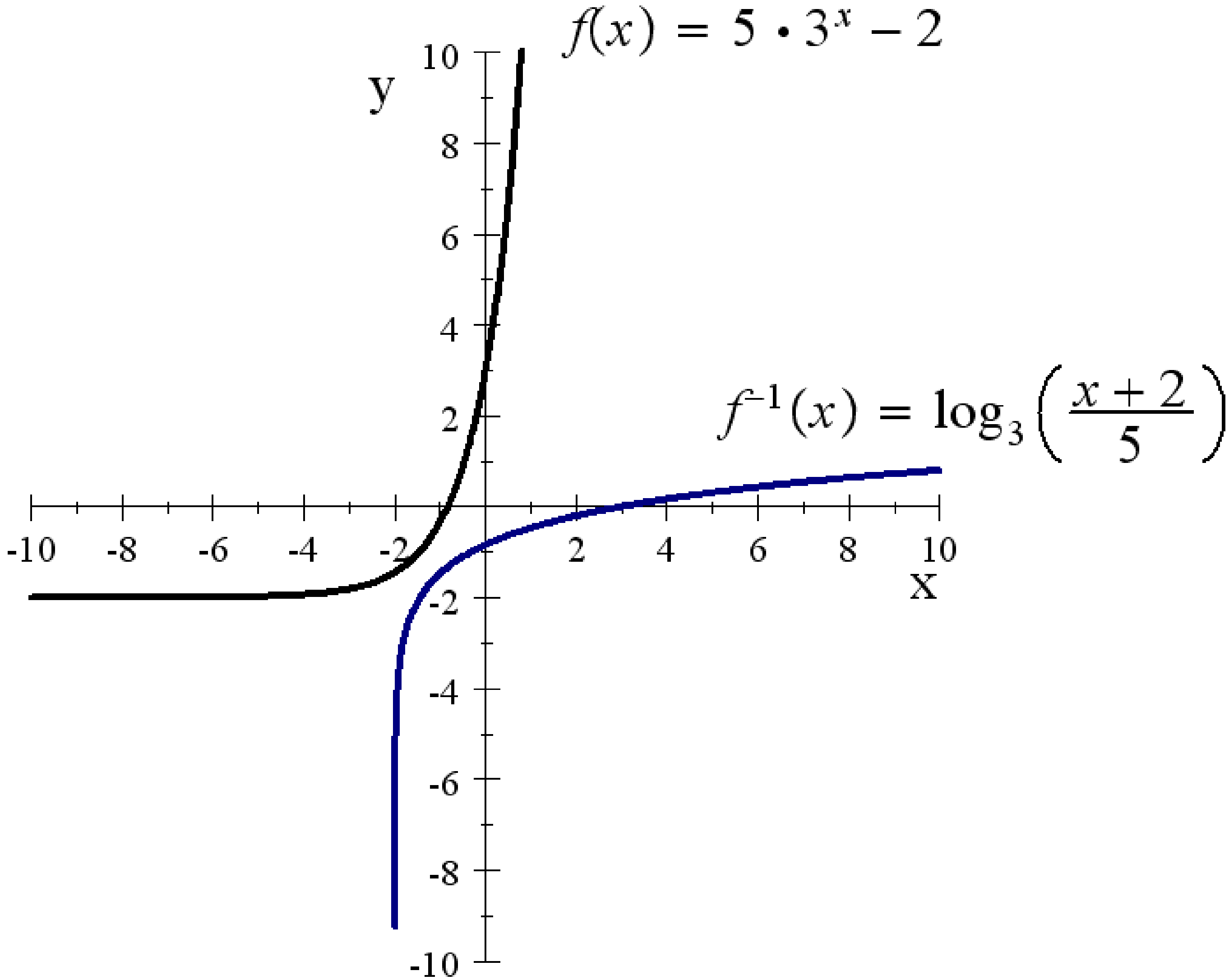
$$\log_3 \mathbf{3}^{y} = \log_3\left(\frac{x+2}{5}\right)$$

$$\mathbf{y} = \log_3\left(\frac{x+2}{5}\right)$$

$$\mathbf{f}^{-1}(x) = \log_3\left(\frac{x+2}{5}\right)$$

$$\mathbf{f}(x) = \mathbf{5 \cdot 3}^{x}\mathbf{-2}$$


y
10
8
6
4
2
-2
-4
-6
-8
-10
x
-10
-8
-6
-4
-2
2
4
6
8
10


$f(x) = 5 \cdot 3^x - 2$
$f^{-1}(x) = \log_3\left(\frac{x+2}{5}\right)$
y
x
10
8
6
4
2
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
2
4
6
8
10